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ON THE FACTORIZATIONS OF ORDINARY LINEAR DIFFERENTIAL OPERATORS

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ABSTRACT. Relations are found between the nonvanishing of certain Wronskians and disconjugacy properties of $L_n y + py = 0$, where $L_n y$ is a disconjugate operator and p is sign definite. The results are then used to show ways in which $L_n y + py$ can be factored.

Introduction. In this paper we investigate possible ways of factoring certain nth order linear differential operators of the form

$$Ly = P_n y^{(n)} + P_{n-1} y^{(n-1)} + \cdots + P_1 y' + P_0 y, \tag{1}$$

where $P_i(x)$ are real valued and $P_n(x) \neq 0$, into products of lower order operators of the same type, where the lower order operators are irreducible.

According to a well-known result of Pólya [**6**], the operator (1) can be factored into a product of first order operators on an interval I if and only if it is disconjugate there. Consequently, we will investigate (1) when it is not disconjugate.

Other work on the factoring of (1) can be found in [**1**, **7**, **8**, and **9**]. We will utilize the following result on factoring due to Zettl [**8**].

THEOREM 1. *Suppose $1 \leq k < n$. The operator* (1) *has a factorization $L = RQ$ where R and Q are of the same type with Q of order k and R of order $n-k$ on an interval I if and only if there are k solutions $y_1, \ldots, y_k$ of $Ly = 0$ which satisfy $\omega(y_1, \ldots, y_k) \neq 0$ on I, where $\omega(y_1, \ldots, y_k)$ is the Wronskian of $y_1, \ldots, y_k$. Further, Q has the representation*

$$Qy = \omega(y_1, \ldots, y_k, y).$$

1. In this section we will list some of the main definitions and results that will be used later for the equations $Ly = 0$, where

$$Ly = L_n y + p(x) y. \tag{2}$$

Throughout, we will assume

(3) $p(x)$ is continuous, real valued and strictly of one sign, and L_n is the disconjugate nth order differential operator

$$L_n y = \rho_n(\rho_{n-1}, \ldots, (\rho_1(\rho_0 y)' \cdots)'$$

with $\rho_i > 0$ and $\rho_i \in C^{n-i}$. We let $L_0 y = \rho_0 y$, $L_i y = \rho_i(L_{i-1} y)'$, $i = 1, \ldots, n$, and call $L_0 y, \ldots, L_n y$ quasi-derivatives of y.

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DEFINITION 1 (ELIAS [**2**]). Let $\sigma(C_0, \ldots, C_n)$ denote the number of sign changes in the sequence $C_0, \ldots, C_n$ of nonzero numbers. Then for a solution $y \not\equiv 0$ of

$$L_n y + p(x) y = 0 \tag{4}$$

and a point x we define

$$S(y, x^+) = \lim_{t \to x^+} \sigma(L_0 y(t), -L_i y(t), \ldots, (-1)^n L_n y(t))$$

and

$$S(y, x^-) = \lim_{t \to x^-} \sigma(L_0 y(t), L_1 y(t), \ldots, L_n y(t)).$$

Let $a \leq x_1 \leq \cdots \leq x_r \leq b$ be the zeros of the quasi-derivatives $L_0 y, L_1 y, \ldots,$ $L_{n-1} y$ of a solution y of (4) in $[a, b]$, where the same $x_i = c$ is used to denote zeros of two different quasi-derivatives $L_j y$ and $L_k y$ if and only if $L_j y(c) = L_k y(c)$ implies either $L_l y(c) = 0$ for all $j \leq l \leq k$ or $L_l y(c) = 0$ for all $k \leq l \leq n-1$ and $0 \leq l \leq j$. With $n(x_i)$ denoting the number of consecutive (with $L_0 y$ following $L_{n-1} y$) quasi-derivatives which vanish at x_i, and $\langle q \rangle$ denoting the greatest even integer not greater than q, we state the following theorem due to Elias [**2**].

THEOREM 2. *Every solution y of* (4) *satisfies the condition*

$$N(y) \equiv S(y, a^+) + S(y, b^-) + \sum_{a < x_t < b} \langle n(x_t) \rangle \leq n. \tag{5}$$

Moreover, $S(y, b^-)$ and $n - S(y, a^+)$ are both even if $p(x) < 0$ and both odd if $p(x) > 0$. If $N(y) = n$, then $L_{t+1} y$ has exactly one sign change between two consecutive zeros of $L_t y$ in $[a, b]$. In addition, $L_{t+1} y$ changes sign before the first zero of $L_t y$ in $(a, b]$ if and only if $\operatorname{sgn}[L_{t+1} y(a + \varepsilon)] = \operatorname{sgn}[L_t y(a + \varepsilon)]$, *and this sign change is unique. The situation is similar near the endpoint b.*

DEFINITION 2. Equation (4) is $(k, n-k)$ disconjugate on $[a, b]$ provided there is no solution y with zeros of order at least k and $n-k$ at a and b, respectively. Equation (4) is $(k, n-k)$ disconjugate on $[a, +\infty)$ if it is $(k, n-k)$ disconjugate on $[a, b]$ for all $a < b < \infty$.

In the discussion that follows, we will let $\gamma < n$ be an integer such that $n - \gamma$ is odd for $p(x) < 0$ and even for $p(x) > 0$. Hence by Theorem 2, (4) will be $(\gamma, n-\gamma)$ disconjugate.

We will give our results using quasi-derivatives rather than derivatives of y. Thus

$$W(y_1, y_2, \ldots, y_r)(x) = \begin{vmatrix} L_0 y_1(x) & \cdots & L_0 y_r(x) \\ L_1 y_1(x) & \cdots & L_1 y_r(x) \\ \vdots & & \vdots \\ L_{r-1} y_1(x) & \cdots & L_{r-1} y_r(x) \end{vmatrix}.$$

We will say y has a zero of order r at a if $L_0 y(a) = \cdots L_1 y(a) = \cdots = L_{r-1} y(a) = 0$. Theorem 1 holds and will be used with W replacing ω.

2. In this section we prove the existence of a basis for the solution space of (4) with certain properties that will be used later.

THEOREM 3. *There are linearly independent solutions y_i of* (4) *for* $i = 0, 1, \dots, n-1$ *with the following properties*:

1. y_i *has a zero of multiplicity exactly* i *at* $x = a$.

2. *If* $b < \infty$, *then* y_i *has a zero of multiplicity at least* $n-1-[i+(1+(-1)^{i-\gamma})/2]$ *at* $x = b$.

3. $z \in \operatorname{span}[y_{\gamma+2r}, \dots, y_{\gamma+2s+1}]$ *for* $0 \le r \le s < (n-\gamma-1)/2$ *implies* $\gamma+2r+1 \le S(z,x^+) \le \gamma+2s+1$ *and* $n-(\gamma+2s+1) \le S(z,x^-) \le n-(\gamma+2r+1)$ *for* $x \in (a,b)$, *where* $b \le \infty$.

4. $W(y_{\gamma+2r}, \dots, y_{\gamma+2s+1})(x) \neq 0$ *for* $x \in (a,b)$.

PROOF. At first we will assume $b < \infty$. In that case, let $y_{\gamma+2j}$ be the nontrivial solution of (4) satisfying

$$\text{(i)} \qquad \begin{aligned} L_i y_{\gamma+2j}(a) &= 0 \quad \text{for } i = 0, \dots, \gamma+2j-1,\ \gamma+2j+1, \\ L_i y_{\gamma+2j}(b) &= 0 \quad \text{for } i = 0, \dots, n-\gamma-2j-3. \end{aligned}$$

Let $y_{\gamma+2j+1}$ be the nontrivial solution of (4) satisfying

$$\text{(ii)} \qquad \begin{aligned} L_i y_{\gamma+2j+1}(a) &= 0 \quad \text{for } i = 0, \dots, 2j+\gamma, \\ L_i y_{\gamma+2j+1}(b) &= 0 \quad \text{for } i = 0, \dots, n-\gamma-2j-3. \end{aligned}$$

In any case define y_0 and y_{n-1} as in (i) or (ii), depending on whether $n-\gamma$ is even or odd. Clearly y_i has a zero of multiplicity i at a and satisfies condition (2). If y_i has a zero of multiplicity greater than i at a for $i = \gamma+2j$ or $i = \gamma+2j+1$, then according to Theorem 2

$$n \ge S(y_i, a^+) + S(y_i, b^-) \ge (\gamma+2j+3) + (n-\gamma-2j-2) = n+1.$$

Thus the y_i satisfy conditions (1), (2) and are linearly independent.

Let $z \in \operatorname{span}[y_{\gamma+2r}, \dots, y_{\gamma+2s+1}]$. Since $S(z,x^+)$ is a nondecreasing function of x which assumes values of the form $\gamma+2j+1$ for j an integer and since z has a zero at a of multiplicity at least $\gamma+2r$, it follows that $\gamma+2r+1 \le S(z,x^+)$ for $x \ge a$. Since $S(z,x^-)$ is nonincreasing and z has a zero at b of multiplicity at least $n-2s-\gamma-1$, it follows that $S(z,x^-) \ge n-2s-\gamma-1$ for $x \le b$. If $\bar{x} \in (a,b)$ is such that $L_i z(\bar{x}) \neq 0$ for $i = 0, \dots, n-1$, then

$$\gamma+2r+1 \le S(z,\bar{x}^+) = n - S(z,\bar{x}^-) \le n - [n-2s-\gamma-1] = 2s+\gamma+1.$$

Also

$$n-2s-\gamma-1 \le S(z,\bar{x}^-) = n - S(z,\bar{x}^+) \le n-(\gamma+2r+1).$$

Since such points are dense in (a,b), property (3) follows.

Suppose there is an $x^* \in (a,b)$ such that $W(y_{\gamma+2r}, \dots, y_{\gamma+2s+1})(x^*) = 0$. Then there is a nontrivial $z \in \operatorname{span}(y_{\gamma+2r}, \dots, y_{\gamma+2s+1})$ with a zero of multiplicity $2s-2r+2$ at x^*. Thus

$$n \ge S(z,a^+) + \langle n(x^*) \rangle + S(z,b^-) \ge (\gamma+2r+1)+(2s-2r+2)+(n-2s-\gamma-1) = n+2.$$

Thus property (4) follows.

To prove the theorem for the case $b = +\infty$, we first apply the above arguments to the functions $y_i(x,n)$ on the interval $[a,n]$. Then using standard compactness arguments, we let $y_i(x) = \lim_{j\to\infty} y_i(x,m_j)$ where $m_j \to \infty$. Conditions (1), (3) and (4) are easily shown to hold for the functions y_i.

3. If (4) is $(\gamma+1, n-\gamma-1)$ disconjugate on $[a,b]$ ($[a,+\infty)$), then according to **[2]** $y_{\gamma+1}$ (given in Theorem 3) has no zeros in (a,b) ($(a,+\infty)$). In this section we extend that result to show that the odd order Wronskian of $y_{\gamma+1}, y_{\gamma+2}, \ldots, y_{\gamma+2k+1}$ for $0 \le k \le (n-\gamma-1)/2$ has no zeros in (a,b) ($(a,+\infty)$).

We will let $\{y_i : 0 \le i \le n-1\}$ be solutions of (4) satisfying the conditions of Theorem 3. For $s \in (a,b)$ we define

$$U(x,t) = \begin{vmatrix} L_0 y_\gamma(s) & \cdots & L_0 y_{\gamma+2k+1}(s) \\ \vdots & & \vdots \\ L_{2k} y_\gamma(s) & & L_{2k} y_{\gamma+2k+1}(s) \\ & & \\ L_\gamma y_\gamma(x) & & L_\gamma y_{\gamma+2k+1}(x) \end{vmatrix}. \tag{6}$$

THEOREM 4. *Suppose for some $s_0 \in (a,b)$, $U(a,s_0) = 0$. Then the function $x(s)$ defined by $U(x,s) = 0$, where $x(s_0) = a$, is such that $dx/ds|_{s=s_0} > 0$.*

PROOF. Define a solution $u_2(x,s)$ of (4) in x by

$$u_2(x,s) \equiv \begin{vmatrix} L_0 y_\gamma(s) & \cdots & L_0 y_{\gamma+2k+1}(s) \\ \vdots & & \vdots \\ L_{2k} y_\gamma(s) & & L_{2k} y_{\gamma+2k+1}(s) \\ & & \\ y_\gamma(x) & & y_{\gamma+2k+1}(x) \end{vmatrix}. \tag{7}$$

Suppose

$$L_{\gamma+1} u_2(a,s_0) = 0.$$

Then there is a solution v_1 of (4) in $\operatorname{span}\{y_\gamma, \ldots, y_{\gamma+2k+1}\}$ with zeros of multiplicities γ, $2k+1$, and $n-\gamma-2k-2$ at a, s_0, and b, respectively. Moreover, $L_{\gamma+1}v_1(a) = 0$. Since $W(a,s_0) = 0$, there is a solution v_2 of (4) in $\operatorname{span}\{y_\gamma, \cdots, y_{\gamma+2k+1}\}$ with zeros of multiplicities $\gamma+1$, $2k+1$, and $n-\gamma-2k-2$ at a, s_0, and b, respectively. If $v_1 = v_2$, then v_2 has a zero of multiplicity $\gamma+2$ at a. Thus by Theorem 2

$$n \ge S(v_2, a^+) + \langle n(s_0) \rangle + S(v_2, b^-) \ge (\gamma+3) + 2k + [n - (\gamma+2k+1)] = n+2.$$

Thus $v_1 \ne v_2$. But in that case there is a linear combination v of v_1 and v_2 with a zero of multiplicity $2k+2$ at s_0. Thus

$$n \ge S(v, a^+) + \langle n(s_0) \rangle + S(v, b^-) \ge (\gamma+1) + (2k+2) + [n - (\gamma+2k+1)] = n+2.$$

Hence $L_{\gamma+1}u_2(a,s_0) \ne 0$. Since $\partial U(a,s_0)/\partial x = L_{\gamma+1}U_2(a,s_0)/\rho_{\gamma+1}(a)$, it follows by the Implicit Function Theorem that $x(s)$ is uniquely defined and

$$\frac{dx}{ds} = -\frac{\partial U}{\partial s} \bigg/ \frac{\partial U}{\partial x}.$$

To show that dx/ds is positive at $s = s_0$, we define

$$u_1(x,s) \equiv \left| \begin{matrix} L_0 y_\gamma(s) & \cdots & L_0 y_{\gamma+2k+1}(s) \\ \vdots & & \vdots \\ L_{2k-1} y_\gamma(s) & & L_{2k-1} y_{\gamma+2k+1}(s) \\ y_\gamma(x) & & y_{\gamma 2k+1(x)} \\ L_\gamma y_\gamma(a) & \cdots & L_\gamma y_{\gamma+2k+1}(a) \end{matrix} \right] \tag{8}$$

and

$$(9)\qquad u_3(x,s) \equiv \begin{vmatrix} L_0 y_\gamma(s) & \cdots & L_0 y_{\gamma+2k+1}(s) \\ \vdots & & \vdots \\ L_{2k-1} y_\gamma(s) & & L_{2k-1} y_{\gamma+2k+1}(s) \\ y_\gamma(x) & & y_{\gamma+2k+1}(x) \\ L_{\gamma+1} y_\gamma(a) & \cdots & L_{\gamma+1} y_{\gamma+2k+1}(a) \end{vmatrix}$$

An application of (5) shows that any solution of (4) satisfying the boundary conditions of $u_2(x,s)$ is essentially unique. But since $U(a,s_0)=0$, $u_1(x,s_0)$ also satisfies those boundary conditions. Hence

$$(10)\qquad u_1(x,s_0) = k u_2(x,s_0).$$

Applying (5) to $u_3(x,s)$ we see that

$$(11)\qquad L_\gamma u_3(a,s) \neq 0.$$

Otherwise

$$n \geq S(u_3,a^+) + \langle n(s)\rangle + S(u_3,b^-) \geq (\gamma+3) + 2k + (n-\gamma-2k-1) = n+2.$$

Also

$$(12)\qquad L_{2k} u_3(s_0,s_0) \neq 0.$$

For if $L_{2k}u_3(s_0,s_0)=0$, then $L_{2k+1}u_3(s_0,s_0)\neq 0$. Otherwise (4) of Theorem 3 is violated. For the same reason

$$(13)\qquad kL_{2k+1}u_1(s_0,s_0) = L_{2k+1}u_2(s_0,s_0) \neq 0.$$

Thus, there is a linear combination of u_2 and u_3, say z, such that $L_i z(s_0,s_0)=0$ for $i=0,1,\ldots,2k+1$, which again violates (4) of Theorem 3.

The Wronskian

$$(14)\qquad \omega(L_i(u_2(x,s_0)), L_i(u_3(x,s_0))) \neq 0 \quad \text{for } x\in(a,s_0),\ i=0,\ldots,n-1.$$

Otherwise, there is a linear combination z of u_2 and u_3 such that $L_i z$ has a double zero at s_1 with $a<s_1<s_0$. Thus by (5)

$$\begin{aligned} n &\geq S(z,\dot{a}^+) + \langle n(s_1)\rangle + \langle n(s_0)\rangle + S(z,b^-) \\ &\geq (\gamma+1) + 2 + (2k) + (n-\gamma-2k-1) = n+2. \end{aligned}$$

We next show that if $u_1(x,s_0)$ has t zeros on (a,s_0), then $u_3(x,s_0)$ has $t+1$ zeros on (a,s_0). By (14) the zeros of $u_3(x,s_0)$ and $u_2(x,s_0)=ku_1(x,s_0)$ separate on (a,s_0). Thus it is enough to show that if s_1 and s_2 are two consecutive zeros of $u_1(x,s_0)$ with either $s_1=a$ or $s_2=s_0$ on $[a,s_0]$, then $u_3(x,s_0)$ has a zero on (s_1,s_2). Suppose $u_3(x,s_0)\neq 0$ for $x\in(s_1,s_2)$. Then $h(x)\equiv u_1(x,s_0)/u_3(x,s_0)$ is continuous on (s_1,s_2). Now $u_1(x,s_0)$ has a zero of order exactly $\gamma+1$ at a, and by (9) and (11) $u_3(x,s_0)$ has a zero of order exactly γ. At $x=s_0$, $u_1(x,s_0)$ and $u_3(x,s_0)$ have zeros of order exactly $2k+1$ and $2k$, respectively. Thus, defining $h(s_1)=h(s_2)=0$, we see by l'Hospital's rule that $h(x)$ is continuous on $[s_1,s_2]$. Since $h(x)\neq 0$ for $x\in(s_1,s_2)$, h must have an extreme point at $\bar{s}\in(s_1,s_2)$ at which $(h)'(\bar{s})=0$. It follows that

$$u_3(\bar{s},s_0)L_1u_1(\bar{s},s_0) - u_1(\bar{s},s_0)L_1u_3(\bar{s},s_0) = 0.$$

Thus $z(x) \equiv u_3(x, s_0)h(\bar{s}) - u_1(x, s_0)$ is such that $L_0 z(\bar{s}) = L_1 z(\bar{s}) = 0$, which is not possible by (14). Thus $u_3(x, s_0)$ has a zero in (s_1, s_2).

By (11) $L_\gamma u_3(a, s_0) \neq 0$. Thus assume, without loss of generality, $L_\gamma u_3(a, s_0) > 0$. Since $L_i u_3(a, s_0) = 0$ for $i = 0, \ldots, \gamma - 1$, it follows that $L_i u_3(a^+, s_0) > 0$ for $i = 0, \ldots, \gamma$. Since by (8) and (9) $L_{\gamma+1} u_1(a, s_0) = -L_\gamma u_3(a, s_0)$, it follows that $L_{\gamma+1} u_1(a, s_0) < 0$. Since $L_i u_1(a, s_0) = 0$ for $i = 0, \ldots, \gamma$, it follows that $L_i u_1(a^+, s_0) < 0$ for $i = 0, \ldots, \gamma + 1$. Suppose $u_1(x, s_0)$ has t zeros, which by (10) and (13) are necessarily simple, in (a, s_0). Then $(-1)^{t+1} u_1(s_0^-, s_0) > 0$. Since $L_i u_1(s_0, s_0) = 0$ for $i = 0, \ldots, 2k$, it follows that

$$(-1)^{t+1+i} L_i u_1(s_0^-, s_0) > 0 \quad \text{for } i = 0, \ldots, 2k + 1. \tag{15}$$

Since $u_3(x, s_0)$ has $t + 1$ simple zeros in (a, s_0), then $(-1)^{t+1} u_3(s_0^-, s_0) > 0$. Since $L_i u_3(s_0, s_0) = 0$ for $i = 0, 1, \ldots, 2k - 1$, it follows that

$$(-1)^{t+1+i} L_i u_3(s_0^-, s_0) > 0 \quad \text{for } i = 0, \ldots, 2k. \tag{16}$$

By (7) and (9) $L_{\gamma+1} u_2(a, s_0) = L_{2k} u_3(s_0, s_0)$. Now

$$\frac{\partial U(a, s_0)}{\partial s} = \frac{L_{2k+1} u_1(s_0, s_0)}{\rho_{2k+1}(s_0)} \quad \text{and} \quad \frac{\partial U(a, s_0)}{\partial x} = \frac{L_{\gamma+1} u_2(a, s_0)}{\rho_{\gamma+1}(a)} = \frac{L_{2k} u_3(s_0, s_0)}{\rho_{\gamma+1}(a)}.$$

Thus

$$\frac{\partial U(a, s_0)/\partial s}{\partial U(a, s_0)/\partial x} = \frac{(-1)^{t+2k+2} L_{2k+1} u_1(s_0, s_0)/\rho_{2k+1}(s_0)}{(-1)(-1)^{t+2k+1} L_{2k} u_3(s_0, s_0)/\rho_{\gamma+1}(a)} < 0$$

by (15), (16), (12) and (13). Hence $dx/ds|_{s=s_0} > 0$.

We define

$$D(x) \equiv \begin{vmatrix} L_0 y_{\gamma+1}(x) & \cdots & L_0 y_{\gamma+2k+1}(x) \\ \vdots & & \vdots \\ L_{2k} y_{\gamma+1}(x) & \cdots & L_{2k} y_{\gamma+2k+1}(x) \end{vmatrix}. \tag{17}$$

THEOREM 5. *There is an $s_0 \in (a, b)$ such that $D(s_0) = 0$ if and only if an $U(a, s_0) = 0$.*

PROOF. Since y_i has a zero of order exactly i at $x = a$, expanding $U(a, s_0)$ by the last row we have $U(a, s_0) = -L_\gamma y_\gamma(a) D(s_0)$.

To show a connection between the zeros of $D(x)$ and disconjugacy properties of (4), we need some information about the distribution of zeros of $L_i u_2(x, s)$.

THEOREM 6. *Let $s \in (a, b)$ and $u_2(x, s)$ be given by (7). Then*
(i) *there exists a first zero x_i of $L_i u_2(x, s)$ in (a, s) for $i = 1, \ldots, \gamma$,*
(ii) $x_\gamma < x_{\gamma-1} < \cdots < x_1$,
(iii) *the simple zeros of $L_j u_2(x, s)$ are differentiable functions of s, and*
(iv) *if $D(s_0) = 0$ for $s_0 \in (a, b)$ ($D(x)$ given by (17)) and if $s > s_0$, then there is $x_0 \in (a, s)$ such that $u_2(x_0, s) = 0$.*

PROOF. Since $L_j u_2(a, s) = 0$ for $j = 0, \ldots, \gamma - 1$, if $L_j u_2(s, s) = 0$ for $j = 0, \ldots, \gamma - 1$, then (i) follows from Rolle's Theorem. Suppose $r < \gamma - 1$ is the largest integer such that $L_j u_2(s, s) = 0$ for $j = 0, \ldots, r$. Then (i) holds for $i = r + 1, \ldots, \gamma$ by applying Rolle's Theorem $i - r$ times. For $i = 1, \ldots, r$ (i) follows from a single application of Rolle's Theorem.

Since $L_t u_2(a,s) = 0$ implies $L_t u_2(x,s)$ and $L_{t+1}u_2(x,s)$ have the same sign on a right neighborhood of a, $L_i(u_2(a,s)) = 0$ for $i = 1, \ldots, \gamma - 1$, and $N(u_2(x,s)) = n$, (ii) follows by Theorem 2.

Part (iii) follows directly from the definition (7) of $u_2(x,s)$ and the Implicit Function Theorem.

Let x_i be the zero of $L_i u_2(x, s^*)$, for $s^* < s_0$, given by (i) for $i = 1, 2, \ldots, \gamma$. By Theorem 4, x_γ cannot exit (a,s) through a as s increases from s^*. Also x_1 cannot exit (a,s) through s; otherwise $u_2(x,s)$ will have a zero of order at least $2k+2$ at s, which implies $N(u_2(x,s)) > n$. By (14) $L_i u_2(x,s)$ cannot have a multiple zero in (a,s). Thus the order of the zeros given by (ii) must be maintained as s increases from s^*. Consequently none of the zeros of (i) can leave (a,s) as s increases from s^*.

By Theorem 4 and Theorem 5, a zero t_γ of $L_\gamma u_2(x,s)$ enters (a,s) through a as s increases through s_0. By Theorem 2, $L_{t+1}u_2(x,s)$ has exactly one sign change between two consecutive zeros of $L_t u_2(x,s)$ for $x \in [a,b]$. Thus we conclude $L_{\gamma-1}u_2(x,s)$ must have a zero, $t_{\gamma-1}$, between t_γ and x_γ. By (ii) $t_{\gamma-1} \neq x_{\gamma-1}$ repeatedly applying Theorem 2 and (ii); we conclude that $u_2(x_0, s) = 0$ for some $x_0 \in (a,s)$.

THEOREM 7. *Suppose there is an* $s_0 \in (a,b)$ $(s_0 \in (a, +\infty))$ *such that* $D(s_0) = 0$ (*where* $D(x)$ *is defined by* (17)). *Then* (4) *is not* $(\gamma+1, n-\gamma-1)$ *disconjugate on* $[a,b]$ $([a,+\infty])$.

PROOF. We will first prove the theorem for the finite case. Let $y(x,s) = \alpha(s)u_2(x,s)$ be such that $\sum_{i=0}^{n-1} L_i^2 y(a,s) = 1$. Then by standard compactness arguments there is a sequence $\{s_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} s_i = b$ and $\lim_{i\to\infty} y(x,s_i) = z(x)$ is a nontrivial solution of (4) and convergence is uniform on $[a,b]$. It follows that $z(x)$ has zeros of order at least γ at a and $n-\gamma-1$ at b. By Theorem 6, for each $s \in (s_0, b)$, $u_2(x,s)$, and thus $y(x,s)$, has a zero in (a,s). Letting $x(s)$ denote the zero of $y(x,s)$ in (a,s), there is a subsequence of $\{s_i\}$, say $\{s_{i_k}\}$, such that $\lim_{k\to\infty} x(s_{i_k}) = x^* \in [a,b]$. Further $z(x^*) = 0$. Now $x^* \neq b$; otherwise z has a zero of order at least $n-\gamma$ at b, and (4) would have an $(\gamma, n-\gamma)$ interval of oscillation, which is impossible. If $x^* = a$, then z has a zero of order at least $\gamma+1$ at a, and thus (4) is not $(\gamma+1, n-\gamma-1)$ disconjugate.

If $x^* \in (a,b)$, define

$$z(x,s) = \begin{vmatrix} L_0 z_1(a) & \cdots & L_0 z_n(a) \\ \vdots & & \vdots \\ L_{\gamma-1} z_1(a) & & L_{\gamma-1} z_n(a) \\ & & \\ L_0 z_1(s) & & L_0 z_n(s) \\ \vdots & & \vdots \\ L_{n-\gamma-2} z_1(s) & & L_{n-\gamma-2} z_n(s) \\ & & \\ z_1(x) & \cdots & z_n(x) \end{vmatrix}$$

where $z_1, \ldots, z_n$ is a basis for the solution space of (4). Since z has zeros of order γ and $n-\gamma-1$ at a and b, respectively, and such solutions are essentially unique, it follows that $z(x,b) = kz(x)$. Thus $z(x^*, b) = 0$. Since $N(z(x,s)) = n$, it follows that the zeros of $z(x,s)$ in (a,s) are simple and thus as in Theorem 6 are differentiable functions of s. Let $x(s)$ be the simple zero of $z(x,s)$ so that $x(b) = x^*$. For s close to a, $z(x,s)$ has no zeros in (a,s). Now $x(s)$ cannot enter (a,s) through s as s increases to b since a $(\gamma, n-\gamma)$ interval of oscillation is impossible. Thus $x(s)$ must enter (a,s) through a, and thus (4) is not $(\gamma+1, n-\gamma-1)$ disconjugate on $[a,b]$.

For the infinite case, we consider the functions $y_i(x,m)$ where

$$y_i(x) = \lim_{m\to\infty} y_i(x,m)$$

as in Theorem 3. Letting

$$D_m(x) \equiv \begin{vmatrix} L_0 y_{\gamma+1}(x,m) & \cdots & L_0 y_{\gamma+2k+1}(x,m) \\ \vdots & & \vdots \\ L_{2k} y_{\gamma+1}(x,m) & \cdots & L_{2k} y_{\gamma+2k+1}(x,m) \end{vmatrix},$$

we have

$$\lim_{m\to\infty} D_m(x) = D(x) \equiv \begin{vmatrix} L_0 y_{\gamma+1}(x) & \cdots & L_0 y_{\gamma+2k+1}(x) \\ \vdots & & \vdots \\ L_{2k} y_{\gamma+1}(x) & \cdots & L_{2k} y_{\gamma+2k+1}(x) \end{vmatrix} \tag{18}$$

Further, since $y_i(x,m)$ and its quasi-derivatives converge uniformly on compact intervals, $D_m(x)$ converges uniformly to $D(x)$ on compact intervals. If $D(s_0) = 0$, then $D'(s_0) \neq 0$. Otherwise there is a solution $u \in \operatorname{span}\{y_{\gamma+1}, \ldots, y_{\gamma+2k+1}\}$ with a zero of multiplicity $2k+2$ at s_0 or $\gamma+2$ at a. But either alternative is impossible by Theorem 3 and Theorem 2. Thus since $D(x)$ changes signs at s_0, there is a large m such that $D_m(x) = 0$ for some $x \in (a,m)$. Thus from the finite case (4) is not $(\gamma+1, n-\gamma-1)$ disconjugate on $[a,m]$ and hence not on $[a,+\infty)$.

THEOREM 8. *If (4) is not $(\gamma+1, n-\gamma-1)$ disconjugate on $[a,b]$ and if b is not a $(\gamma+1, n-\gamma-1)$ conjugate point of a, then $D(x)$ must have a zero in (a,b).*

PROOF. By Elias [**2**, Theorem 3], it is enough to show that $D(x) \neq 0$ on (a,b) implies there is a solution of (4) with zeros of multiplicities γ and $n-\gamma-1$ at a and b, respectively, with no zeros in (a,b).

There is an $s > a$ such that $u_2(x,s) \neq 0$ for $x \in (a,s)$. As s increases from a, a zero cannot enter (a,s) through s. Otherwise for some $s_1 \in (a,b)$, $L_{2k+1}u_2(s_1,s_1) = 0$ and by (5)

$$\begin{aligned} N(u_2(x,s_1)) &\geq S(u_2,a^+) + S(u_2,b^-) + \langle n(s_1)\rangle \\ &\geq (\gamma+1) + [n-(\gamma+2k+1)] + 2k+2 = n+2. \end{aligned}$$

Thus if $u_2(x,s)$ has a zero in (a,s), it must enter through a as s increases. But that means $L_\gamma u_2(a,s_2) = 0$ for some $s_2 \in (a,b)$. But by Theorem 5 and (7) that is possible only when $D(s_2) = 0$. Thus for every $s \in (a,b)$, $u_2(x,s) \neq 0$ for $x \in (a,s)$. Letting $y(x,s) = \alpha(s)u_2(x,s)$ as in the proof of Theorem 7 and letting s approach b along a suitable sequence, we obtain a solution $z(x)$ of (4) with zeros of multiplicities at least γ and $n-\gamma-1$ at a and b, respectively. But since b is not a $(\gamma+1, n-\gamma-1)$ conjugate point of a, the multiplicities of the zeros above must be exact. If $z(x_1)=0$ for $x_1 \in (a,b)$, then $L_1 z(x_1)=0$ also. But by Theorem 2 that is not possible. Thus $z(x)$ has no zeros on (a,b).

An immediate corollary of Theorem 8 is

COROLLARY 1. *If $D(x) \neq 0$ for $x \in (a,b)$, then* (4) *is disconjugate on $[a,c]$ for every $c<b$.*

Defining $D(x)$ by (17) using the solutions $y_i(x)$ for $i=0,\dots,n-1$ of (4) given by Theorem 3 with $b=+\infty$, we obtain the following version of Theorem 8 for the infinite interval.

THEOREM 9. *If $D(x) \neq 0$ for $x \in (a,+\infty)$, then any solution u of* (4) *with $S(u,x^+)=\gamma+1$ eventually, is nonoscillatory.*

PROOF. We define $u_2(x,s)$ by (7), using the solutions $y_i(x)$ for $i=0,\dots,n-1$ of (4) given by Theorem 3 with $b=+\infty$. Now $S(u,x^+)$ is an integer valued nondecreasing function, while $S(u,x^-)$ is nonincreasing. According to Theorem 3, $S(u_2(x,s),x^+) \geq \gamma+1$, while $S(u_2(x,s),x^-) \geq n-(\gamma+2k+1)$ for $x \in (a,+\infty)$. Letting $x_1 \in [a,s)$ and $x_2 \in (s,+\infty)$, we have by (5) that

$$\begin{aligned} n &\geq S(u_2(x,s),x_1^+) + \langle n(s)\rangle + S(u_2(x,s),x_2^-) \\ &= (\gamma+1+j)+2k+[n-(\gamma+2k+1)+l] \quad \text{for } j,l \geq 0. \end{aligned}$$

Thus $n \geq n+j+l$, which implies $j=l=0$. Thus $S(u_2(x,s),x_1^+)=\gamma+1$, while $S(u_2(x,s),x_2^-)=n-(\gamma+2k+1)$.

There is an $s>a$ such that $u_2(x,s) \neq 0$ for $x \in (a,s)$. Otherwise by taking the limit as s approaches a and normalizing as in Theorem 7, we obtain a solution u of (4) with a zero of multiplicity $\gamma+2k+2$ at a with $S(u,x^-)=n-(\gamma+2k+1)$ for $x>a$, which violates (5). Thus exactly as in the finite case (Theorem 8) $u_2(x,s) \neq 0$ for $x \in (a,s)$ and $s \in (a,+\infty)$. Taking the limit as s goes to $+\infty$ and normalizing as in Theorem 7, we obtain a solution u of (4) such that $S(u,x^+)=\gamma+1$ for $x \in [a,+\infty)$ with no zeros in $(a,+\infty)$. According to Elias [**2**], this implies the desired conclusion.

4. In this section we will show that $(\gamma+2k+1, n-\gamma-2k-1)$ disconjugacy of (4) on $[a,b]$ ($[a,+\infty)$) implies the Wronskian of $y_\gamma,\dots,y_{\gamma+2k}$ for $0 \leq k \leq (n-\gamma-1)/2$ has no zeros in (a,b) ($(a,+\infty)$), where $y_i(x)$ are the solutions of (4) given by Theorem 3.

The results here will parallel those of §3. Hence the proofs will be omitted.

We define functions H and T to play the role of U and D of §3 in the following way:

$$H(x,s) \equiv \begin{vmatrix} L_0 y_\gamma(s) & \cdots & L_0 y_{\gamma+2k+1}(s) \\ \vdots & & \vdots \\ L_{2k} y_\gamma(s) & & L_{2k} y_{\gamma+2k+1}(s) \\ & & \\ L_{n-\gamma-2k-2} y_\gamma(x) & \cdots & L_{n-\gamma-2k-2} y_{\gamma+2k+1}(x) \end{vmatrix} \tag{19}$$

and

$$T(x) \equiv \begin{vmatrix} L_0 y_\gamma(x) & \cdots & L_0 y_{\gamma+2k}(x) \\ \vdots & & \vdots \\ L_{2k} y_\gamma(x) & \cdots & L_{2k} y_{\gamma+2k}(x) \end{vmatrix} \tag{20}$$

In Theorem 11 and below we will need the added condition that

$$L_{n-\gamma-2k-2} y_{\gamma+2k}(b) = 0.$$

If b is not a $(\gamma + 2k + 1, n - \gamma - 2k - 1)$ conjugate point of a, there is no loss in generality in making that assumption. If b is a $(\gamma + 2k + 1, n - \gamma - 2k - 1)$ conjugate point of a, then $L_{n-\gamma-2k-2} y_{\gamma+2k+1}(b) = 0$, and we can interchange the role of $y_{\gamma+2k}$ and $y_{\gamma+2k+1}$ in the statement of the theorems of this section. Thus throughout this section we will assume $L_{n-\gamma-2k-2} y_{\gamma+2k}(b) = 0$.

THEOREM 10. *Suppose for some $s_0 \in (a, b)$, $H(b, s_0) = 0$. Then the function $x(s)$ defined by $H(x, s) = 0$, where $x(s_0) = b$, is such that $dx/ds|_{s=s_0} > 0$.*

Using the definitions of H and T, we have the following analog of Theorem 5.

THEOREM 11. *There is an $s_0 \in (a, b)$ such that $T(s_0) = 0$ if and only if $H(b, s_0) = 0$.*

THEOREM 12. *Set $s \in (a, b)$ and let $u_2(x, s)$ be given by (7). Then*
(i) *there is a last zero x_i of $L_i u_2(x, s)$ in (s, b) for $i = 1, 2, \ldots, n - \gamma - 2k - 2$,*
(ii) *$x_1 < x_2 < \cdots < x_{n-\gamma-2k-2}$,*
(iii) *if $T(s_0) = 0$ for $s_0 \in (a, b)$ and if $s < s_0$, then $u_2(x_0, s) = 0$ for some $x_0 \in (s, b)$.*

As in §3, we have

THEOREM 13. *Suppose there is an $s_0 \in (a, b)$ $((a, +\infty))$ such that $T(s_0) = 0$. Then (4) is not $(\gamma + 2k + 1, n - \gamma - 2k - 1)$ disconjugate on $[a, b]$ $([a, +\infty))$.*

THEOREM 14. *If (4) is not $(\gamma + 2k + 1, n - \gamma - 2k - 1)$ disconjugate on $[a, b]$ and if b is not a $(\gamma + 2k + 1, n - \gamma - 2k - 1)$ conjugate point of a, then $T(x)$ must have a zero in (a, b).*

5. We now apply the results of §§2, 3 and 4 to give some specific factorizations of (2). We will assume the following disconjugacy conditions for (2) on I, where $I = [a, b]$ or $[a, +\infty)$.

(A) (2) is not $(\gamma + 1 + 2k, n - \gamma - 1 - 2k)$ disconjugate on $[a, c]$ for some $c < b$, for $k = 0, \ldots, j_0$;

(2) is $(i, n - i)$ disconjugate on I for $i < \gamma + 1$ or $i > \gamma + 1 + 2j_0$.

As shown in [**4**] for some ray $[b, +\infty)$ and in [**3**] for finite intervals, our assumption is exactly what happens when $L_n y \equiv y^{(n)}$. For example, if n is even but not divisible by 4, $p(x) > 0$, $I = [a, +\infty)$ and $y'' + [x^{(n-2)}/(n - 2)!]p(x)y = 0$ is disconjugate, then (2) is $(1, n - 1)$ disconjugate on I [**5**]. However, if

$$y'' + \left[\left(\int_t^\infty (s - t)p(s)\,ds\right) \Big/ \left(\frac{n}{2} - 2\right)! \left(\frac{n}{2} - 1\right)!\right] y = 0$$

is oscillatory, then (2) is not $(n/2, n/2)$ disconjugate on I [**5**]. Thus by [3] there is a γ such that (2) is not $(\gamma + 1 + 2k, n - \gamma - 1 - 2k)$ disconjugate on I for $k = 0, \ldots, n/2 - \gamma - 1$ but is $(i, n-i)$ disconjugate on I for $i < \gamma + 1$ or $i > n - \gamma - 1$.

The factorization for (2) given in our next theorem can easily be obtained by using methods similar to Theorem 3. However, using theorems of §§3 and 4, we will show other factorizations are possible.

THEOREM 15. *If the operator* (2) *satisfies assumption* (A), *then it admits a factorization on the interior of* I *of the form*

$$L = L_0 \cdots L_{\gamma-1} Q_0 \cdots Q_{j_0} L_{\gamma+2j_0+2} \cdots L_{n-1}, \tag{21}$$

where the L_i *are first and the* Q_i *are second order operators of the form* (1) *with each* Q_i *irreducible.*

PROOF. Let $y_0, y_1, \ldots, y_{n-1}$ be as in Theorem 3. Then

$$W(y_i, y_{i+1}, \ldots, y_{n-1})(x_0) = 0$$

for $x_0 > a$ if and only if there is a z in span$\{y_i, y_{i+1}, \ldots, y_{n-1}\}$ with a zero of multiplicity $n - i$ at x_0. But since z has a zero of multiplicity i at a, it follows that $W(y_i, y_{i+1}, \ldots, y_{n-1})(x_0) = 0$ for $x_0 > a$ if and only if (2) is not $(i, n-1)$ disconjugate on I. Thus repeatedly using Theorem 1, it follows that (2) admits factorization (21). To see that Q_l is irreducible, we observe that $z_1 \equiv W(y_{\gamma+2l+1}, y_{\gamma+2l+2}, \ldots, y_{n-1})$ and $z_2 \equiv W(y_{\gamma+2l}, y_{\gamma+2l+2}, \ldots, y_{n-1})$ are independent solutions of $Q_l y = 0$. Since (2) is not $(2l + 1 + \gamma, n - 2l - 1 - \gamma)$ disconjugate on I, it follows by the first part of the proof that z_1 must have a zero at some $x_1 > a$. But since the order of the zero of any other solution z of $Q_l y = 0$ is less than that of z_1 at a, z must have a zero at some x_2 where $a < x_2 < x_1$. Thus Q_l fails to be disconjugate and hence by the classical result of Pólya [**5**], cannot be factored.

Although (21) seems to be a natural way to factor (2), our next theorem shows that it is not the only way.

THEOREM 16. *If the operator* (2) *satisfies assumption* (A), *then it admits a factorization on the interior of* I *of the form*

$$L = L_0 \cdots L_{\gamma-1} L_{\gamma+2j_0+2} \cdots L_{n-1} Q_0 \cdots Q_{j_0}, \tag{22}$$

where the L_i *are first and the* Q_i *second order operators of the form* (1) *with each* Q_i *irreducible.*

PROOF. Let $y_0, y_1, \ldots, y_{n-1}$ be a basis for the solution space of $Ly = 0$ as in Theorem 3. Then as in the first part of Theorem 15, $W(y_i, y_{i+1}, \ldots, y_{n-1})(x) \neq 0$ for $x \in \operatorname{int} I$ and for $i = 1, \ldots, \gamma$ since (2) is $(i, n - i)$ disconjugate on I. Thus applying Theorem 1 repeatedly, $L = L_0 \cdots L_{\gamma-1} Q$, where $y_\gamma, y_{\gamma+1}, \ldots, y_{n-1}$ is a fundamental set of solutions for $Qy = 0$. Now applying Theorem 3 for even order Wronskians and Theorem 13 for odd order Wronskians, $W(y_\gamma, y_{\gamma+1}, \ldots, y_j)(x) \neq 0$ for $x \in \operatorname{int} I$ for $x \in \operatorname{int} I$ since (2) is $(\gamma + j, n - \gamma - j)$ disconjugate for $j \geq \gamma + 1 + 2j_0$. Again applying Theorem 1 repeatedly, $Q = L_{\gamma+2j_0+2} \cdots L_{n-1} P$, where $y_\gamma, y_{\gamma+1}, \ldots, y_{\gamma+2j_0+l}$ is a fundamental set of solutions for $Py = 0$. Applying Theorem 3, we see

$$W(y_{\gamma+2r}, y_{\gamma+2r+1}, \ldots, y_{\gamma+2j_0+1})(x) \neq 0$$

for $x \in \operatorname{int} I$ and $r = 1, \ldots, j_0$. Thus again using Theorem 1, $P = Q_0 \cdots Q_{j_0}$, where each Q_i is second order. To see that Q_l is irreducible, we observe that

$$z_1 = W(y_{\gamma+2l+1}, y_{\gamma+2l+2}, \ldots, y_{\gamma+2j_0+1})$$

and

$$z_2 = W(y_{\gamma+2l}, y_{\gamma+2l+2}, y_{\gamma+2l+3}, \ldots, y_{\gamma+2j_0+1})$$

are independent solutions of $Q_l y = 0$. Since (2) is not $(2l+1+\gamma, n-2l-1-\gamma)$ disconjugate, by the corollary to Theorem 8, z_1 must have a zero at $x_1 \in \operatorname{int} I$. But since the order of the zero of any other solution z of $Q_l y = 0$ is less than that of z_1 at a, z must have a zero at some x_2 where $a < x_2 < x_1$. Thus Q_l fails to be disconjugate and hence cannot be factored.

COROLLARY. *If the operator* (2) *satisfies assumption* (A), *then it admits a factorization on the interior of I of the form*

$$L = PQ, \tag{23}$$

where P is disconjugate and every solution of $Qy = 0$ oscillates.

PROOF. Let $Q = Q_0 \cdots Q_{j_0}$. Then $y_\gamma, y_{\gamma+1}, \ldots, y_{\gamma+2j+1}$ is a fundamental set of solutions for $Qy = 0$. Thus by Theorem 3 every solution of $Qy = 0$ oscillates. Letting $P = L_0 \cdots L_{\gamma-1} L_{\gamma+2j+2} \cdots L_{n-1}$, it follows that P is disconjugate.

Applying the results of §§3 and 4, it is clear that many other factorizations are possible.

EXAMPLE. Let $Ly \equiv y^{(4)} - y$, $L_1 y \equiv -y'/2e^x$, $L_2 y \equiv e^x[y' - y]$ and $Q_1 y = -2y'' + 2y' - 4y$. Then $Ly = L_1 \circ Q_1 \circ L_2(y)$ as in Theorem 15. But if $L_3 y \equiv 2e^x y' - 2e^x y$ and $Q_2 y = -y'' - y$, then $Ly = L_1 \circ Q_2 \circ L_3 y$ as in Theorem 16.

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